Which action for brane worlds?

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Abstract: In his pioneering work on singular shells in general relativity, Lanczos had derived jump conditions across energy-momentum carrying hypersurfaces from the Einstein equation with codimension 1 sources. However, on the level of the action, the discontinuity of the connection arising from a codimension 1 energy-momentum source requires to take into account two adjacent space-time regions separated by the hypersurface.
The purpose of the present note is to draw attention to the fact that Lanczos' jump conditions can be derived from an Einstein action but not from an Einstein–Hilbert action.

1 Introduction

Recently, a particular class of cosmological models commonly denoted as brane-world models attracted a lot of attention. These models are essentially based on two assumptions:

• Our universe may be described as a timelike four-dimensional submanifold Σ of a $(1+3+n)$–dimensional bulk space-time $\mathcal{S}$, with $n \geq 1$ additional spacelike dimensions.

• The additional spacelike dimensions can only be probed by gravity and eventually some non-standard matter degrees of freedom, but standard model particles cannot leave our observable universe Σ.

Predecessors of this kind of cosmological scenarios rely on dynamical binding mechanisms for low energy matter to an effectively four-dimensional submanifold, which has some finite but small extension in the transverse dimensions. Dynamical mechanisms for explaining such scenarios have been proposed already by Akama [1], by Rubakov and Shaposhnikov [2], by Visser [3], and by Gibbons and Wiltshire [4], and the corresponding transversally "thick" universes have also attracted much attention recently [5], see also [6, 7] and references there.

The other extreme, which was partly motivated from string theory, consists of 3-branes Σ which have no transverse extension at all and are strictly codimension-n submanifolds [8, 9, 10, 11, 12, 13, 14]. Here the confinement of matter to Σ is not necessarily a dynamical phenomenon, but imposed axiomatically through the assumption that matter degrees contribute only a hypersurface integral over Σ to an action S which also contains bulk terms for gravity and eventually a few other bulk degrees of freedom. Such an axiomatic distinction between hypersurface and bulk degrees of freedom may seem strange at first sight, but *a priori* there is nothing mathematically inconsistent with it, and so there is no *a priori* reason to rule out such scenarios[1].

It has been mentioned already that such $(1+3)$–dimensional submanifolds go by the name 3-branes, but referring to the old literature on singular timelike 3-manifolds in $1+3$ dimensions (e.g. [32, 33]) another appropriate term would be hypersurface layers. Layers denote hypersurfaces with discontinuous extrinsic curvature across the hypersurface due to the presence of energy-momentum on the hypersurface.

The purpose of this note is to draw attention to the fact that an Einstein action instead of an Einstein–Hilbert action does yield the same jump conditions across a hypersurface layer as the Einstein equation.

[1]Whether or not we find *a posteriori* reasons is another story, of course. But this can only be clarified by investigations of (post-)Newtonian limits [15, 16, 17, 18, 19] and of cosmological implications of these models [11, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

The following notation is used:

$$f(x)|_{x=a} = f(a), \qquad [f(x)]_{x=a}^{x=b} = f(b) - f(a).$$

It is helpful to start with a toy model from electrodynamics before we address the Einstein–Hilbert action in section 3:

2 A toy model: Electrodynamics with planar codimension 1 sources

Electrodynamics in $1+3$ dimensions with codimension 1 sources located on the plane $x^3 = 0$ is described by an action

$$S = -\frac{1}{4}\int dt \int d^2\mathbf{x} \int_{-\infty}^{\infty} dx^3 F^{\mu\nu}F_{\mu\nu} + \int dt \int d^2\mathbf{x}\,(\mathbf{j}\cdot\mathbf{A} + j^0 A_0)\Big|_{x^3=0},$$

where $d^2\mathbf{x} = dx^1 dx^2$ and all vectors are two-dimensional vectors: $\mathbf{j}\cdot\mathbf{A} = j^1 A_1 + j^2 A_2$. Without much ado we write down the equations of motion which follow from $\delta S = 0$:

$$\partial_\mu F^{\mu\nu} = -j^\nu \delta(x^3), \tag{1}$$

implying in particular

$$\lim_{\epsilon\to+0}\left[F^{3\nu}\right]_{x^3=-\epsilon}^{x^3=\epsilon} = -\,j^\nu\big|_{x^3=0}. \tag{2}$$

Of course, this can be confirmed from a more careful evaluation of the variation of S:

$$S = -\frac{1}{4}\lim_{\epsilon\to+0}\left(\int dt \int d^2\mathbf{x} \int_{-\infty}^{-\epsilon} dx^3 F^{\mu\nu}F_{\mu\nu} + \int dt \int d^2\mathbf{x} \int_{\epsilon}^{\infty} dx^3 F^{\mu\nu}F_{\mu\nu}\right)$$

$$+ \int dt \int d^2\mathbf{x}\,(\mathbf{j}\cdot\mathbf{A} + j^0 A_0)\Big|_{x^3=0},$$

whence

$$\delta S = \lim_{\epsilon\to+0}\left(\int dt \int d^2\mathbf{x} \int_{-\infty}^{-\epsilon} dx^3 \delta A_\nu \partial_\mu F^{\mu\nu} + \int dt \int d^2\mathbf{x} \int_{\epsilon}^{\infty} dx^3 \delta A_\nu \partial_\mu F^{\mu\nu}\right)$$

$$- \lim_{\epsilon\to+0}\int dt \int d^2\mathbf{x}\left[\delta A_\nu F^{3\nu}\right]_{x^3=\epsilon}^{x^3=-\epsilon} + \int dt \int d^2\mathbf{x}\left(\delta\mathbf{A}\cdot\mathbf{j} + \delta A_0 j^0\right)\Big|_{x^3=0}.$$

Therefore, (1) and the jump condition (2) indeed imply $\delta S = 0$.
However, this does not work with the Einstein–Hilbert action:

3 A first *Ansatz* for the action in brane models

For simplicity I pretend that I can cover a $(1+4)$–dimensional space-time by a single coordinate patch x^M which is Gaussian close to the brane: $x^0 = t$, x^j $(1 \leq j \leq 3)$ are tangential to the world-volume of the brane, and x^5 is normal: $g_{\mu 5} = 0$ on the brane, $0 \leq \mu \leq 3$. It is known that the geodesic distance from the brane provides such a coordinate system locally, whence the brane is localized at $x^5 = 0$. If this coordinate system cannot be extended to all of the five-dimensional space-time (which is what one expects), we have to glue together several patches with appropriate transition functions to formulate action principles. However, the difficulty that we encounter with the Einstein–Hilbert action is related only to the boundary conditions across the brane, and therefore we write the Einstein–Hilbert *Ansatz* for the brane action as

$$S_{EH} = \int dt \int d^3\mathbf{x} \int_{-\infty}^{\infty} dx^5 \sqrt{-g} \left(\frac{m^3}{2} R - \Lambda \right) + \int dt \int d^3\mathbf{x}\, \mathcal{L} \Big|_{x^5=0}, \tag{3}$$

where we assume that the brane Lagrangian $\mathcal{L}$ contains no genuine gravitational terms: Derivatives of the metric appear in $\mathcal{L}$ only through covariant derivatives on fermions and eventually massive vector fields.

One might expect an Einstein equation to emerge from (3):

$$R_{MN} - \left(\frac{1}{2} R - \frac{\Lambda}{m^3} \right) g_{MN} = -\frac{2}{m^3 \sqrt{-g}} \delta(x^5) \frac{\delta \mathcal{L}}{\delta g^{MN}}. \tag{4}$$

However, a naive derivation of (4) from (3) would have to assume continuity of normal derivatives across the brane, in *a posteriori* contradiction to (4).

To clarify this and to reveal which equations would really follow from stationarity of S_{EH}, we write it more carefully as

$$S_{EH} = \lim_{\epsilon \to +0} \left(\int dt \int d^3\mathbf{x} \int_{-\infty}^{-\epsilon} dx^5 \sqrt{-g} \left(\frac{m^3}{2} R - \Lambda \right) \right. \tag{5}$$

$$\left. + \int dt \int d^3\mathbf{x} \int_{\epsilon}^{\infty} dx^5 \sqrt{-g} \left(\frac{m^3}{2} R - \Lambda \right) \right) + \int dt \int d^3\mathbf{x}\, \mathcal{L} \Big|_{x^5=0}.$$

Variation of the metric then yields

$$\delta S_{EH} = \frac{m^3}{2} \lim_{\epsilon \to +0} \left(\int dt \int d^3\mathbf{x} \int_{-\infty}^{-\epsilon} dx^5 \sqrt{-g} \delta g^{MN} \left(R_{MN} - \frac{1}{2} g_{MN} R + \frac{\Lambda}{m^3} g_{MN} \right) \right.$$

$$\left. + \int dt \int d^3\mathbf{x} \int_{\epsilon}^{\infty} dx^5 \sqrt{-g} \delta g^{MN} \left(R_{MN} - \frac{1}{2} g_{MN} R + \frac{\Lambda}{m^3} g_{MN} \right) \right) \tag{6}$$

$$
+\frac{m^3}{2}\lim_{\epsilon\to+0}\int dt\int d^3\mathbf{x}\left[\sqrt{-g}\left(g^{MN}\delta\Gamma^5{}_{MN}-g^{5N}\delta\Gamma^M{}_{MN}\right)\right]_{x^5=\epsilon}^{x^5=-\epsilon}
$$

$$
+\int dt\int d^3\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^5=0}
$$

$$
=\frac{m^3}{2}\lim_{\epsilon\to+0}\left(\int dt\int d^3\mathbf{x}\int_{-\infty}^{-\epsilon}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\right.
$$

$$
\left.+\int dt\int d^3\mathbf{x}\int_{\epsilon}^{\infty}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\right)
$$

$$
+\frac{m^3}{4}\lim_{\epsilon\to+0}\int dt\int d^3\mathbf{x}[\sqrt{-g}(3\delta g^{\mu\nu}g^{55}\partial_5 g_{\mu\nu}-2\delta g^{\mu 5}g^{55}\partial_\mu g_{55}
$$

$$
-\delta g^{55}g^{\mu\nu}\partial_5 g_{\mu\nu}+2g^{55}g_{\mu\nu}\partial_5\delta g^{\mu\nu})]_{x^5=\epsilon}^{x^5=-\epsilon}+\int dt\int d^3\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^5=0}.
$$

The jump conditions following from $\delta S_{EH}=0$ are incompatible with the jump conditions following from (4) (see eq. (9) below). In fact, $\delta S_{EH}=0$ would even require a traceless energy-momentum tensor on the brane if $\delta\mathcal{L}/\delta g^{55}=0$.
In an attempt to infer the jump conditions following from (4) from an action principle, we will consider the Einstein action next:

4 An Einstein action for brane models

The Einstein action proves more suitable in boundary models [16], and may also be better adapted to brane models:

$$
S_E=\lim_{\epsilon\to+0}\left(\int dt\int d^3\mathbf{x}\int_{-\infty}^{-\epsilon}dx^5\sqrt{-g}\left(\frac{m^3}{2}g^{MN}\left(\Gamma^K{}_{LM}\Gamma^L{}_{KN}-\Gamma^K{}_{KL}\Gamma^L{}_{MN}\right)-\Lambda\right)\right.
$$

$$
\left.+\int dt\int d^3\mathbf{x}\int_{\epsilon}^{\infty}dx^5\sqrt{-g}\left(\frac{m^3}{2}g^{MN}\left(\Gamma^K{}_{LM}\Gamma^L{}_{KN}-\Gamma^K{}_{KL}\Gamma^L{}_{MN}\right)-\Lambda\right)\right) \tag{7}
$$

$$
+\int dt\int d^3\mathbf{x}\,\mathcal{L}\bigg|_{x^5=0}
$$

$$
=S_{EH}-\frac{m^3}{2}\lim_{\epsilon\to+0}\int dt\int d^3\mathbf{x}\left[\sqrt{-g}\left(g^{MN}\Gamma^5{}_{MN}-g^{5N}\Gamma^M{}_{MN}\right)\right]_{x^5=\epsilon}^{x^5=-\epsilon},
$$

with variation under changes of the metric:

$$\begin{aligned}
\delta S_E = &\frac{m^3}{2}\lim_{\epsilon\to+0}\Big(\int dt\int d^3\mathbf{x}\int_{-\infty}^{-\epsilon}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\\
&+\int dt\int d^3\mathbf{x}\int_{\epsilon}^{\infty}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\Big) \qquad (8)\\
&-\frac{m^3}{4}\lim_{\epsilon\to+0}\int dt\int d^3\mathbf{x}\Big[2\sqrt{-g}\left(\delta g^{MN}\Gamma^5{}_{MN}-\delta g^{5N}\Gamma^M{}_{MN}\right)\\
&-\sqrt{-g}\delta g^{MN}g_{MN}\left(g^{KL}\Gamma^5{}_{KL}-g^{5L}\Gamma^K{}_{KL}\right)\Big]_{x^5=\epsilon}^{x^5=-\epsilon}+\int dt\int d^3\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^5=0}\\
= &\frac{m^3}{2}\lim_{\epsilon\to+0}\Big(\int dt\int d^3\mathbf{x}\int_{-\infty}^{-\epsilon}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\\
&+\int dt\int d^3\mathbf{x}\int_{\epsilon}^{\infty}dx^5\sqrt{-g}\delta g^{MN}\left(R_{MN}-\frac{1}{2}g_{MN}R+\frac{\Lambda}{m^3}g_{MN}\right)\Big)\\
&+\frac{m^3}{4}\lim_{\epsilon\to+0}\int dt\int d^3\mathbf{x}\Big[\sqrt{-g}\delta g^{\mu\nu}\left(g^{55}\partial_5 g_{\mu\nu}-g_{\mu\nu}g^{\alpha\beta}g^{55}\partial_5 g_{\alpha\beta}\right)\\
&-\sqrt{-g}\delta g^{5\mu}\left(g^{55}\partial_\mu g_{55}-g^{\alpha\beta}\partial_\mu g_{\alpha\beta}\right)\Big]_{x^5=\epsilon}^{x^5=-\epsilon}+\int dt\int d^3\mathbf{x}\,\delta g^{MN}\frac{\delta\mathcal{L}}{\delta g^{MN}}\bigg|_{x^5=0}.
\end{aligned}$$

$\delta S_E=0$ thus yields a five-dimensional Einstein space in the bulk

$$R_{MN}=\frac{2\Lambda}{3m^3}g_{MN},$$

and the five-dimensional analog of the Lanczos equations [32, 33]:

$$\lim_{\epsilon\to+0}\left[\partial_5 g_{\mu\nu}\right]_{x^5=-\epsilon}^{x^5=\epsilon}=-\frac{2}{m^3}\sqrt{g_{55}}\left(T_{\mu\nu}-\frac{1}{3}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}\right)\bigg|_{x^5=0}, \qquad (9)$$

i.e. exactly the equations that one infers from the Einstein equation[2] (4). Here the brane energy-momentum tensor is defined via

$$T_{\mu\nu}=-\frac{2}{\sqrt{-\det(g_{\alpha\beta})}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\bigg|_{x^5=0}.$$

Another advantage of the Einstein action is the disappearance of δg^{55} on the brane, implying that the Einstein action complies with the usual assumption that the brane Lagrangian $\mathcal{L}$ depends only on the induced metric on the brane.

[2] A further equation on the brane appears if the brane is a boundary of space-time [16]: In this case the term $\sim\delta g^{5\mu}$ cannot be cancelled by continuity across the brane and requires

$$g^{55}\partial_\mu g_{55}\big|_{x^5=0}=g^{\alpha\beta}\partial_\mu g_{\alpha\beta}\big|_{x^5=0},$$

i.e. $g_{55}\sim-\det(g_{\alpha\beta})$ on a boundary.

5 Conclusion

The jump conditions following from the Einstein equation with brane sources imply stationarity of the Einstein action with brane sources, but not of the Einstein–Hilbert action with brane sources. Furthermore, stationarity of the Einstein action complies with brane Lagrangians $\mathcal{L}$ which do not depend on the normal component of the metric.

One might be concerned about diffeomorphism invariance since the Einstein action is only invariant under $IGL(5)$ transformations. However, we have seen that stationarity of the Einstein action is equivalent to the fully covariant Einstein equation (4) (remembering that x^5 is a geodesic distance, i.e. a well-defined geometric object). Therefore, besides the numerical value of the action itself, no classical results inferred from the use of an Einstein action depend on the coordinate system.

Note added: An important reference on early investigations in brane cosmology is Chamblin and Reall [34]. These authors had also already recognized the difficulty with the Einstein–Hilbert action for thin branes and added a Gibbons–Hawking term to cure the problem.

References

[1] K. Akama, *Pregeometry*, in: Gauge Theory and Gravitation, edited by K. Kikkawa, N. Nakanishi and H. Nariai, Springer–Verlag, Berlin 1983, pp. 267–271.

[2] V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. 125B (1983) 136.

[3] M. Visser, Phys. Lett. 159B (1985) 22.

[4] G.W. Gibbons, D.L. Wiltshire, Nucl. Phys. B287 (1987) 717.

[5] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B429 (1998) 263.

[6] C. Csáki, J. Erlich, T.J. Hollowood, Y. Shirman, Nucl. Phys. B581 (2000) 309.

[7] C. Grojean, Phys. Lett. B479 (2000) 273.

[8] P. Hořava, E. Witten, Nucl. Phys. B460 (1996) 506, B475 (1996) 94.

[9] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436 (1998) 257.

[10] A. Lukas, B.A. Ovrut, K.S. Stelle, D. Waldram, Phys. Rev. D59 (1999) 086001, Nucl. Phys. B552 (1999) 246.

[11] P. Binétruy, C. Deffayet, D. Langlois, Nucl. Phys. B565 (2000) 269.

[12] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

[13] K.R. Dienes, E. Dudas, T. Gherghetta, Nucl. Phys. B567 (2000) 111.

[14] K. Behrndt, M. Cvetič, Phys. Lett. B475 (2000) 253.

[15] J. Garriga, T. Tanaka, Phys. Rev. Lett. 84 (2000) 2778.

[16] R. Dick, D. Mikulović, Phys. Lett. B476 (2000) 363.

[17] W. Mück, K.S. Viswanathan, I.V. Volovich, hep-th/0002132.

[18] I.Ya. Aref'eva, M.G. Ivanov, W. Mück, K.S. Viswanathan, I.V. Volovich, hep-th/0004114.

[19] I. Giannakis, H.-c. Ren, hep-th/0007053.

[20] T. Nihei, Phys. Lett. B465 (1999) 81.

[21] J.M. Cline, C. Grojean, G. Servant, Phys. Rev. Lett. 83 (1999) 4245.

[22] P. Kraus, JHEP 9912 (1999) 011.

[23] A. Yu. Neronov, Phys. Lett. B472 (2000) 273.

[24] P. Binétruy, C. Deffayet, U. Ellwanger, D. Langlois, Phys. Lett. B477 (2000) 285.

[25] D.N. Vollick, hep-th/9911181.

[26] D. Ida, gr-qc/9912002.

[27] M. Cvetič, J. Wang, Phys. Rev. D61 (2000) 124020.

[28] J.M. Cline, hep-ph/0001285.

[29] H. Collins, B. Holdom, hep-ph/0003173v2.

[30] N. Deruelle, T. Doležel, gr-qc/0004021.

[31] H. Stoica, S.-H.H. Tye, I. Wasserman, Phys. Lett. B482 (2000) 205.

[32] K. Lanczos, Annalen Phys. (4. Serie) 74 (1924) 518.

[33] W. Israel, Nuovo Cim. X 44B (1966) 1, (E) X 48B (1967) 463.

[34] H.A. Chamblin, H.S. Reall, Nucl. Phys. B562 (1999) 133.